A Proof of the Explicit Minimal-basis Expansion of Tree Amplitudes in Gauge Field Theory

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ABSTRACT: In last couple years, an important relation (BCJ relation) between color-ordered tree-level scattering amplitudes of gauge theory has inspired many studies. This relation implies that the minimal basis for the color-ordered tree-level amplitudes is $(n-3)!$ and other amplitudes can be expanded into a particular chosen basis. In this paper we will prove the conjectured explicit minimal basis expansion. For this purpose we will write down general BCJ relation of gauge theory by taking the field theory limit of BCJ relation in string theory. Then we prove these general BCJ relations using BCFW on-shell recursion relation. Using these general BCJ relations, we prove the conjectured explicit minimal-basis expansion of gauge theory tree amplitudes inductively.

Contents

1 Introduction

Recently, there are significant progresses about various relations of tree-level scattering amplitudes of gauge theory and gravitational theory. For gauge theory, old result (Kleiss-Kuijf(KK)[1] relation) stated that any color-ordered tree-level amplitude of n gluons can be expanded by a basis with $(n-2)!$ amplitudes. However, this result has been revised after Bern, Carrasco and Johansson conjectured a new highly nontrivial relation(BCJ relation)[2] for gauge theory which significantly reduces the number of basis from $(n-2)!$ to $(n-3)!$[1]. The new discovered BCJ relation plays a very important role for our understanding of another important old result in gravity theory (Kawai-Lewellen-Tye(KLT)[4] relation) which expresses tree-level amplitude of n gravitons by the sum of products of two color-ordered tree-level amplitudes of n gluons with appropriate kinematic factors[5, 6].

These relations have been investigated from the point of view of string theory as well as field theory. From the point of view of string theory, KK relation, BCJ relation[7, 8] and KLT relation[4, 9] are consequences of monodromy relations. After taking the field theory limit in string theory, these relations appear naturally.

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[1]The BCJ relation for amplitudes with gluons coupled to matter is suggested in [3].

Although it is convenience to embed gauge theory and gravity theory into string theory, it is not necessary to do so. In fact, as the consistent consideration, it is desirable to have a pure field theory proof of these facts. The old result (KK relation) has been proved by new color-decomposition [10]. The new discovered fundamental BCJ relation (as well as KK relation) and KLT relation have been proved by Britto-Cachazo-Feng-Witten(BCFW) on-shell recursion relation[18, 19] along the line of S-matrix program[20] in [11, 13] and [14–17][2]. From the point of view of these proofs, the BCJ relation is the bonus relation of the improved vanishing behavior for non-nearby BCFW-deformation [21]. BCJ relation is also the consistent condition for the equivalence of various KLT relations.

BCFW recursion relation[18, 19] is an important tool to calculate and study amplitudes. With BCFW recursion relation, one can construct on-shell tree amplitudes by sub-amplitudes with less external legs

$$M_n = \sum_{\mathcal{I}, \mathcal{J}, \mathbf{h}} \frac{M_{\mathcal{I}}(\widehat{p}_i, \widehat{P}_{\mathcal{I}, \mathcal{J}}^{\mathbf{h}}) M_{\mathcal{J}}(\widehat{p}_j, -\widehat{P}_{\mathcal{I}, \mathcal{J}}^{-\mathbf{h}})}{P_{\mathcal{I}\mathcal{J}}}, \tag{1.1}$$

where the sum is over all possible distributions of external legs with shifted momentum $\widehat{p}_i \in \mathcal{I}$, $\widehat{p}_j \in \mathcal{J}$ and $z_{\mathcal{I}\mathcal{J}}$ indicates the location of deformation where inner propagator is on-shell. Via discussions on complex analysis, this relation exists in theories with proper vanishing behavior $M(z \to \infty) = 0$ under BCFW-deformation. Both gauge theory and gravity theory satisfy this condition[19, 21, 22]. BCFW recursion relation can also be written down with nontrivial boundary contributions as considered in [23][3].

Because its importance, there have been lots of works on BCJ relation. From our point of view, we think that BCJ relation can be understood from following two levels of meaning. The first level of meaning is that there is a set of constraint equations which reduce the number of independent amplitudes from $(n-2)!$ to $(n-3)!$. The second level of meaning is the explicit minimal-basis expansion of amplitudes, i.e., how other amplitudes can be written down as the linear combination of basis with explicit expression of coefficients. For the first level of claim, in string theory, these constraints come from monodromy relations when we do the contour deformation and they contain not only the fundamental BCJ relation, but also other relations which we will call "general BCJ relations"[7, 8]. The fundamental BCJ relation has been proved in field theory by BCFW recursion relation, but the general BCJ relations have not been proved and we will give a proof in this paper. For the second level of claim, until now it is still a conjecture [2] and there is no explicit proof. It is our main purpose in this paper to give such a proof, thus complete the whole claim.

The outline of our paper is following. In section 2 we will first consider the field theory limit of the general BCJ relation in string theory, which provides a set of constraints on gauge theory amplitudes. It reduces the number of the independent amplitudes from $(n-2)!$ to $(n-3)!$. Though the general BCJ relation can be derived from string theory directly, we give also a field theory proof by BCFW recursion

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[2]In [12] BCJ relation is explained from the point of view of Schouten identity.

[3]BCFW recursion relation has been generalized to loop level as well as string theory. A few references can be found, for example, in [24] and[25].

relation in section 3. These general BCJ relations will be used for the proof of the explicit minimal-basis expansion of gauge theory amplitudes in section 4. Finally in section 5 we give a brief conclusion.

2   The field theory limit of the BCJ relation in string theory

Gauge field theory can be embed into open string theory as its massless field limit. String theory often provides many useful information for studying gauge theory. One of such examples is the beautiful proof of KK and BCJ relations[7, 8] in string theory, where monodromy plays a crucial role in the proof. In this section, we will take the field theory limit of the BCJ relation in string theory.

KK relation for open string tree amplitudes is given as[7]

$$
\begin{aligned}
&A_n(\beta_1, ..., \beta_r, 1, \alpha_1, ..., \alpha_s, n) \\
&= (-1)^r \times \mathcal{R}e \left[\prod_{1 \leq i < j \leq r} e^{2i\pi\alpha' k_{\beta_i} \cdot k_{\beta_j}} \sum_{\{\sigma\} \in P(O\{\alpha\} \cup O\{\beta^T\})} \prod_{i=0}^{s} \prod_{j=1}^{r} e^{2i\pi\alpha'(\alpha_i, \beta_j)} A_n(1, \{\sigma\}, n) \right],
\end{aligned}
\tag{2.1}
$$

where we have defined $\alpha_0 = 1$. Our notations are following. $O\{\alpha\}$ means to keep the relative ordering inside the set α while α^T means to take the reversed ordering of set α. Putting together $P(O\{\alpha\} \cup O\{\beta^T\})$ denote all permutations of set $O\{\alpha\} \cup O\{\beta^T\}$ where relative orderings inside set α and set β^T have been preserved. This relation expresses $n = r + s + 2$ point open string tree amplitudes by $(n-2)!$ amplitudes.

BCJ relation for open string tree amplitudes is given as[7]

$$
\mathcal{I}m \left[\prod_{1 \leq i < j \leq r} e^{2i\pi\alpha' k_{\beta_i} \cdot k_{\beta_j}} \sum_{\{\sigma\} \in P(O\{\alpha\} \cup O\{\beta^T\})} \prod_{i=0}^{s} \prod_{j=1}^{r} e^{2i\pi\alpha'(\alpha_i, \beta_j)} A_n(1, \{\sigma\}, n) \right] = 0,
\tag{2.2}
$$

where (α, β) is defined as

$$
(\alpha, \beta) = \begin{cases} k_\alpha \cdot k_\beta \ (x_\beta > x_\alpha) \\ \quad 0 \qquad \text{otherwise} \end{cases}.
\tag{2.3}
$$

Combining with KK-relation (2.1), this relation reduces further the number of independent open string amplitudes from $(n-2)!$ to $(n-3)!$.

Noticing that

$$
\begin{aligned}
&\prod_{1 \leq i < j \leq r} e^{2i\pi\alpha' k_{\beta_i} \cdot k_{\beta_j}} \sum_{\{\sigma\} \in P(O\{\alpha\} \cup O\{\beta^T\})} \prod_{i=0}^{s} \prod_{j=1}^{r} e^{2i\pi\alpha'(\alpha_i, \beta_j)} A_n(1, \{\sigma\}, n) \\
&= \sum_{\{\sigma\} \in P(O\{\alpha\} \cup O\{\beta^T\})} \exp \left(i\pi\alpha' \sum_{1 \leq i < j \leq r} s_{\beta_i \beta_j} + \sum_{i=0}^{s} \sum_{j=1}^{r} i\pi\alpha'(\alpha_i, \beta_j) \right) A_n(1, \{\sigma\}, n) \\
&= \sum_{\{\sigma\} \in P(O\{\alpha\} \cup O\{\beta^T\})} \exp \left(i\pi\alpha' \sum_{i=1}^{r} \sum_{\sigma_J < \sigma_{\beta_i}} s_{\beta_i J} \right) A_n(1, \{\sigma\}, n),
\end{aligned}
\tag{2.4}
$$

where σ_J denotes the position of the open string J in the permutation $\{\sigma\}$ with the convention that the position of particle 1 is defined as $\sigma_1 = 0$, we can take the field theory limit $\alpha' \to 0$ where only massless modes (i.e., gluons) of open string theory are left. The leading contribution of the real part of Eq. (2.1) gives the familiar KK relation for color-ordered gluon amplitudes in field theory

$$A_n(\beta_1, ..., \beta_r, 1, \alpha_1, ..., \alpha_s, n) = (-1)^r \sum_{\{\sigma\} \in P(O\{\alpha\} \cup O\{\beta^T\})} A_n(1, \{\sigma\}, n). \tag{2.5}$$

The leading contribution of the imaginary part of Eq. (2.2) gives

$$\sum_{\{\sigma\} \in P(O\{\alpha\} \cup O\{\beta\})} \sum_{i=1}^{r} \sum_{\sigma_J < \sigma_{\beta_i}} s_{\beta_i J} A_n(1, \{\sigma\}, n) = 0, \tag{2.6}$$

where we have redefined $\{\beta\}^T \to \{\beta\}$ with r-elements. This is nothing, but the BCJ relation for color-ordered gluon amplitudes in field theory. When there is only one element in $\{\beta\}$, it just gives the fundamental BCJ relation, which has been proved in field theory [11–13]. The formula (2.6) is more general in following sense. We can divide remaining $(n-2)$-elements into arbitrary two sets α, β with the number of elements $(n-2-r), r$ respectively and with arbitrary ordering. Because this freedom, remaining $(n-2)$-elements are all same footing.

As we have mentioned, the BCJ relation(2.6) provides further constraints on amplitudes to reduce the number of independent amplitudes from $(n-2)!$ to $(n-3)!$. However, (2.6) provides only a set of constraints on amplitudes, and it does not give the explicit expressions of amplitudes by minimal-basis. To obtain these explicit expressions, one should solve these constraint equations to express any amplitudes by $(n-3)!$ independent ones. However, the general solving is very nontrivial and based on some examples, an explicit minimal-basis expression was conjectured in [2]. In following two sections, we will first give the field theory proof of the constraint equations (2.6), then we will use these constraints to prove (solve) the conjectured explicit minimal-basis expression.

3 The field theory proof of the general BCJ relation

Though the general BCJ relation (2.6) can be derived from the BCJ relation in string theory directly by taking the field theory limit, its pure field theory proof is still desirable. In this section, we will give a field theory proof of the general BCJ relation (2.6) by BCFW recursion relation. To demonstrate our idea of proof, we show an example first.

3.1 An example

The first nontrivial example for the general BCJ relation is the six point amplitude with set $\beta = \{2, 3\}$ and set $\alpha = \{4, 5\}$ and it is given by

$$0 = I_6 = (s_{21} + s_{31} + s_{32})A(1, 2, 3, 4, 5, 6) + (s_{21} + s_{31} + s_{32} + s_{34})A(1, 2, 4, 3, 5, 6)$$

$$+ (s_{21} + s_{31} + s_{32} + s_{34} + s_{35})A(1,2,4,5,3,6) + (s_{21} + s_{24} + s_{31} + s_{34} + s_{32})A(1,4,2,3,5,6)$$

$$+ (s_{21} + s_{24} + s_{31} + s_{34} + s_{32} + s_{35})A(1,4,2,5,3,6)$$

$$+ (s_{21} + s_{24} + s_{25} + s_{31} + s_{34} + s_{35} + s_{32})A(1,4,5,2,3,6) \tag{3.1}$$

To show this is true, there are two possible ways to go. The first way is to use the fundamental BCJ relation recursively[11], while the second way, the BCFW recursion relation[18, 19]. In this paper we will use the second method where p_1 and p_6 are the shifted momenta. Thus the R. H. S. of the above equation is

$$(s_{21} + s_{31} + s_{32}) \left[A(\widehat{1}, 2|3,4,5,\widehat{6}) + A(\widehat{1}, 2, 3|4,5,\widehat{6}) + A(\widehat{1}, 2, 3, 4|5,\widehat{6}) \right]$$

$$+ (s_{21} + s_{31} + s_{32} + s_{34}) \left[A(\widehat{1}, 2|4,3,5,\widehat{6}) + A(\widehat{1}, 2, 4|3,5,\widehat{6}) + A(\widehat{1}, 2, 4, 3|5,\widehat{6}) \right]$$

$$+ (s_{21} + s_{31} + s_{32} + s_{34} + s_{35}) \left[A(\widehat{1}, 2|4,5,3,\widehat{6}) + A(\widehat{1}, 2, 4|5,3,\widehat{6}) + A(\widehat{1}, 2, 4, 5|3,\widehat{6}) \right]$$

$$+ (s_{21} + s_{24} + s_{31} + s_{34} + s_{32}) \left[A(\widehat{1}, 4|2,3,5,\widehat{6}) + A(\widehat{1}, 4, 2|3,5,\widehat{6}) + A(\widehat{1}, 4, 2, 3|5,\widehat{6}) \right]$$

$$+ (s_{21} + s_{24} + s_{31} + s_{34} + s_{32} + s_{35}) \left[A(\widehat{1}, 4|2,5,3,\widehat{6}) + A(\widehat{1}, 4, 2|5,3,\widehat{6}) + A(\widehat{1}, 4, 2, 5|3,\widehat{6}) \right]$$

$$+ (s_{21} + s_{24} + s_{25} + s_{31} + s_{34} + s_{35} + s_{32}) \left[A(\widehat{1}, 4|5,2,3,\widehat{6}) + A(\widehat{1}, 4, 5|2,3,\widehat{6}) + A(\widehat{1}, 4, 5, 2|3,\widehat{6}) \right] . \tag{3.2}$$

where we have used following convention $A(\widehat{1}, 4, 5, 2|3, \widehat{6}) \equiv \frac{\sum_h A_L(\widehat{1},4,5,2,\widehat{P}_{3,6}^h) A_R(-\widehat{P}_{3,6}^{-h}, 3, \widehat{6})}{s_{36}}$. Since the momentum of k_1 in the BCFW expansion of amplitudes is shifted, when we use the BCJ relation for sub-amplitudes A_L or A_R, we should use the kinematic factors $s_{2\widehat{1}}$ and $s_{3\widehat{1}}$ instead of s_{21} and s_{31}. Thus we should write $s_{21} = s_{2\widehat{1}} + (s_{21} - s_{2\widehat{1}})$ etc. Putting it back, above expression can be split into two parts $\mathbb{A}$ and $\mathbb{B}$. $\mathbb{A}$ part contains terms with kinematic factors $s_{2\widehat{1}}$, $s_{3\widehat{1}}$ and $s_{ij}(i, j \neq 1)$, while $\mathbb{B}$ part contains terms with kinematic factors $s_{21} - s_{2\widehat{1}}$ and $s_{31} - s_{3\widehat{1}}$.

The $\mathbb{A}$ part can be rewritten as

$$\mathbb{A} = \left[\left(s_{2\widehat{1}} + s_{3\widehat{P}_{12}} \right) A(\widehat{1}, 2|3,4,5,\widehat{6}) + \left(s_{2\widehat{1}} + s_{3\widehat{P}_{12}} + s_{34} \right) A(\widehat{1}, 2|4,3,5,\widehat{6}) \right.$$

$$+ \left(s_{2\widehat{1}} + s_{3\widehat{P}_{12}} + s_{34} + s_{35} \right) A(\widehat{1}, 2|4,5,3,\widehat{6}) \right] + \left[\left(s_{2\widehat{1}} + s_{3\widehat{1}} + s_{32} \right) A(\widehat{1}, 2, 3|4,5,\widehat{6}) \right]$$

$$+ \left[\left(s_{2\widehat{1}} + s_{3\widehat{1}} + s_{32} \right) A(\widehat{1}, 2, 3, 4|5,\widehat{6}) \right] + \left[\left(s_{2\widehat{P}_{14}} + s_{3\widehat{P}_{14}} + s_{32} \right) A(\widehat{1}, 4|2,3,5,\widehat{6}) \right.$$

$$+ \left(s_{2\widehat{P}_{14}} + s_{3\widehat{P}_{14}} + s_{32} + s_{35} \right) A(\widehat{1}, 4|2,5,3,\widehat{6}) + \left(s_{2\widehat{P}_{14}} + s_{25} + s_{3\widehat{P}_{14}} + s_{35} + s_{32} \right) A(\widehat{1}, 4|5,2,3,\widehat{6}) \right]$$

$$+ \left[\left(s_{2\widehat{1}} + s_{3\widehat{P}_{124}} \right) A(\widehat{1}, 2, 4|3,5,\widehat{6}) + \left(s_{2\widehat{1}} + s_{24} + s_{3\widehat{P}_{142}} \right) A(\widehat{1}, 4, 2|3,5,\widehat{6}) \right.$$

$$+ \left(s_{2\widehat{1}} + s_{3\widehat{P}_{124}} + s_{35} \right) A(\widehat{1}, 2, 4|5,3,\widehat{6}) + \left(s_{2\widehat{1}} + s_{24} + s_{3\widehat{P}_{142}} + s_{35} \right) A(\widehat{1}, 4, 2|5,3,\widehat{6}) \right]$$

$$+ \left[\left(s_{2\widehat{P}_{145}} + s_{3\widehat{P}_{145}} + s_{32} \right) A(\widehat{1}, 4, 5|2,3,\widehat{6}) \right]$$

$$+\Big[\big(s_{2\widehat{1}}+s_{3\widehat{1}}+s_{32}+s_{34}\big)\,A(\widehat{1},2,4,3|5,\widehat{6})+\big(s_{2\widehat{1}}+s_{24}+s_{3\widehat{1}}+s_{34}+s_{32}\big)\,A(\widehat{1},4,2,3|5,\widehat{6})\Big]$$

$$+\Big[\Big(s_{2\widehat{1}}+s_{3\widehat{P}_{1245}}\Big)\,A(\widehat{1},2,4,5|3,\widehat{6})+\Big(s_{2\widehat{1}}+s_{24}+s_{3\widehat{P}_{1425}}\Big)\,A(\widehat{1},4,2,5|3,\widehat{6})$$

$$+\Big(s_{2\widehat{1}}+s_{24}+s_{25}+s_{3\widehat{P}_{1452}}\Big)\,A(\widehat{1},4,5,2|3,\widehat{6})\Big] \tag{3.3}$$

For a given BCFW splitting (i.e., the particular cut, for example $(12|3456)$) we should use the BCJ relation for A_L or A_R sub-amplitudes. For example, the splitting (12) can be grouped as

$$\Big(s_{2\widehat{1}}+s_{3\widehat{P}_{12}}\Big)\,A(\widehat{1},2|3,4,5,\widehat{6})+\Big(s_{2\widehat{1}}+s_{3\widehat{P}_{12}}+s_{34}\Big)\,A(\widehat{1},2|4,3,5,\widehat{6})$$

$$+\Big(s_{2\widehat{1}}+s_{3\widehat{P}_{12}}+s_{34}+s_{35}\Big)\,A(\widehat{1},2|4,5,3,\widehat{6})$$

$$= s_{2\widehat{1}}\Big(A(\widehat{1},2|3,4,5,\widehat{6})+A(\widehat{1},2|4,3,5,\widehat{6})+A(\widehat{1},2|4,5,3,\widehat{6})\Big)$$

$$+s_{3\widehat{P}_{12}}A(\widehat{1},2|3,4,5,\widehat{6})+\Big(s_{3\widehat{P}_{12}}+s_{34}\Big)A(\widehat{1},2|4,3,5,\widehat{6})+\Big(s_{3\widehat{P}_{12}}+s_{34}+s_{35}\Big)A(\widehat{1},2|4,5,3,\widehat{6})$$

$$= 0, \tag{3.4}$$

where we have used the BCJ relation for three point amplitudes and five point amplitudes. Similar arguments can be used to show that whole $\mathbb{A}$ part vanishes.

The $\mathbb{B}$ part can be rewritten as

$$\mathbb{B}=s_{21}\Big[A(1,2,3,4,5,6)+A(1,2,4,3,5,6)+A(1,2,4,5,3,6)+A(1,4,2,3,5,6)$$

$$+A(1,4,2,5,3,6)+A(1,4,5,2,3,6)\Big]$$

$$+s_{31}\Big[A(1,2,3,4,5,6)+A(1,2,4,3,5,6)+A(1,2,4,5,3,6)+A(1,4,2,3,5,6)$$

$$+A(1,4,2,5,3,6)+A(1,4,5,2,3,6)\Big]$$

$$+\oint_{z\neq0}\frac{dz}{z}s_{2\widehat{1}}\Big[A(\widehat{1},2,3,4,5,\widehat{6})+A(\widehat{1},2,4,3,5,\widehat{6})+A(\widehat{1},2,4,5,3,\widehat{6})+A(\widehat{1},4,2,3,5,\widehat{6})$$

$$+A(\widehat{1},4,2,5,3,\widehat{6})+A(\widehat{1},4,5,2,3,\widehat{6})\Big]$$

$$+\oint_{z\neq0}\frac{dz}{z}s_{3\widehat{1}}\Big[A(\widehat{1},2,3,4,5,\widehat{6})+A(\widehat{1},2,4,3,5,\widehat{6})+A(\widehat{1},2,4,5,3,\widehat{6})+A(\widehat{1},4,2,3,5,\widehat{6})$$

$$+A(\widehat{1},4,2,5,3,\widehat{6})+A(\widehat{1},4,5,2,3,\widehat{6})\Big]$$

$$=-\oint_{z=\infty}\frac{dz}{z}s_{2\widehat{1}}\Big[A(\widehat{1},2,3,4,5,\widehat{6})+A(\widehat{1},2,4,3,5,\widehat{6})+A(\widehat{1},2,4,5,3,\widehat{6})+A(\widehat{1},4,2,3,5,\widehat{6})$$

$$+A(\widehat{1},4,2,5,3,\widehat{6})+A(\widehat{1},4,5,2,3,\widehat{6})\Big]$$

$$-\oint_{z=\infty}\frac{dz}{z}s_{3\widehat{1}}\Big[A(\widehat{1},2,3,4,5,\widehat{6})+A(\widehat{1},2,4,3,5,\widehat{6})+A(\widehat{1},2,4,5,3,\widehat{6})+A(\widehat{1},4,2,3,5,\widehat{6})$$

$$+A(\widehat{1}, 4, 2, 5, 3, \widehat{6}) + A(\widehat{1}, 4, 5, 2, 3, \widehat{6})\Big], \qquad (3.5)$$

where $\oint_{z \neq 0}$ means the big enough contour around $z = 0$ but does not include the contribution from pole $z = 0$. Using the KK relation, we can see

$$A(\widehat{1}, 2, 3, 4, 5, \widehat{6}) + A(\widehat{1}, 2, 4, 3, 5, \widehat{6}) + A(\widehat{1}, 2, 4, 5, 3, \widehat{6}) + A(\widehat{1}, 4, 2, 3, 5, \widehat{6})$$
$$+ A(\widehat{1}, 4, 2, 5, 3, \widehat{6}) + A(\widehat{1}, 4, 5, 2, 3, \widehat{6})$$
$$= A(\{3, 2\}, \widehat{1}, \{4, 5\}, \widehat{6}). \qquad (3.6)$$

Since in $A(\{3, 2\}, \widehat{1}, \{4, 5\}, \widehat{6})$, the $\widehat{1}$ and $\widehat{6}$ are not nearby, the $z \to \infty$ behavior[21] is $\frac{1}{z^2}$, the integrals in $\mathbb{B}$ must vanish. Thus finally we proved the BCJ relation for six gluon amplitudes (3.1). It is worth to notice that the method of proof is similar to the one used for the fundamental BCJ relation[11].

In the next subsection, we will extend the proof of this example to the proof of n-particle BCJ relation (2.6) with arbitrary sets α, β.

3.2   General proof

Now let us turn to the field theory proof of the general BCJ relation (2.6). The starting point of the recursive proof is the BCJ relation for three point amplitudes

$$s_{21} A_3(1, 2, 3) = 0. \qquad (3.7)$$

This relation can be seen obviously, since $s_{21} = p_3^2 = 0$.

For the general formula where the momenta of the legs 1 and n are the shifted momenta in the BCFW expansion, we should notice that there are two types of dynamical factors s_{ij}: one contains the shifted momentum p_1 and another, does not. As seen in previous example, these two types should be treated separately when we use BCFW recursion relation to expand amplitudes and when we apply general BCJ relation inductively to sub-amplitudes, where we should use the shifted factors $s_{\beta_i \widehat{1}}$, i.e., we should write $s_{\beta_i 1} = s_{\beta_i \widehat{1}} + \left(s_{\beta_i 1} - s_{\beta_i \widehat{1}}\right)$.

As in the previous example, we can divide the expression into two parts: part $\mathbb{A}$ and the part $\mathbb{B}$. The contribution of part $\mathbb{B}$ with factors $\left(s_{\beta_i 1} - s_{\beta_i \widehat{1}}\right)$ is given as

$$\sum_{\{\sigma\} \in P(O\{\alpha\} \cup O\{\beta\})} \sum_{i=1}^{r} s_{\beta_i 1} A(1, \{\sigma\}, n) - \sum_{\{\sigma\} \in P(O\{\alpha\} \cup O\{\beta\})} \sum_{i=1}^{r} \sum_{\text{All splitting}} s_{\beta_i \widehat{1}} A(\widehat{1}, \{\sigma_L\} | \{\sigma_R\}, \widehat{n})$$

$$= \sum_{i=1}^{r} s_{\beta_i 1} \sum_{\{\sigma\} \in P(O\{\alpha\} \cup O\{\beta\})} A(1, \{\sigma\}, n) + \sum_{i=1}^{r} \sum_{\{\sigma\} \in P(O\{\alpha\} \cup O\{\beta\})} \oint_{big \ z \neq 0} \frac{dz}{z} s_{\beta_i \widehat{1}} A(\widehat{1}, \{\sigma\}, \widehat{n})$$

$$= \sum_{i=1}^{r} \sum_{\{\sigma\} \in P(O\{\alpha\} \cup O\{\beta\})} \left[- \oint_{z=\infty} \frac{dz}{z} s_{\beta_i \widehat{1}} A(\widehat{1}, \{\sigma\}, \widehat{n}) \right]$$

$$= (-1)^{r+1} \sum_{i=1}^{r} \oint_{z=\infty} \frac{dz}{z} s_{\beta_i \widehat{1}} A(\{\beta^T\}, \widehat{1}, \{\alpha\}, \widehat{n}), \tag{3.8}$$

where we have used the KK relation in the last line. Since $\widehat{1}$ and $\widehat{n}$ are not nearby, the boundary behavior is $\frac{1}{z^2}$[21], thus the integral around the infinity vanishes.

The contribution of part $\mathbb{A}$ of (2.6) is given by following BCFW recursion relation as

$$\sum_{\{\sigma\} \in P(O\{\alpha\} \cup O\{\beta\})} \sum_{i=1}^{r} \sum_{\sigma_J < \sigma_{\beta_i}} s_{\beta_i J} \sum_{\text{All splitting}} A_n(\widehat{1}, \{\sigma_L\} | \{\sigma_R\}, \widehat{n})$$

$$= \sum_{i=1}^{r} \sum_{\sigma_J < \sigma_{\beta_i}} s_{\beta_i J} \sum_{\text{All splitting}} \sum_{\{\sigma_L\} \subset P(O\{\alpha_L\} \cup O\{\beta_L\})} \sum_{\{\sigma_R\} \in P(O\{\alpha_R\} \cup O\{\beta_R\})} A_n(\widehat{1}, \{\sigma_L\} | \{\sigma_R\}, \widehat{n})$$

$$= \sum_{\text{All splitting}} \left\{ \left[\sum_{\sigma_L \in P(O\{\alpha_L\} \cup O\{\beta_L\})} \sum_{i=1}^{r_L} \sum_{0 \leq \sigma_J < \beta_{Li}} s_{\beta_{Li} J_L} A_{r_L + s_L + 2}(\widehat{1}, \{\sigma_L\}, -\widehat{P}_{\mathcal{I}}) \right] \times \frac{1}{P_{\mathcal{I}}^2} \right.$$

$$\times \left[\sum_{\{\sigma_R\} \in P(O\{\alpha_R\} \cup O\{\beta_R\})} A_{n - r_L - s_L - 2} A(\widehat{P}_{\mathcal{I}}, \{\sigma_R\}, \widehat{n}) \right]$$

$$+ \left[\sum_{\{\sigma_L\} \in P(O\{\alpha_L\} \cup O\{\beta_L\})} A_{r_L + s_L + 2}(\widehat{1}, \{\sigma_L\}, -\widehat{P}_{\mathcal{I}}) \right] \times \frac{1}{P_{\mathcal{I}}^2}$$

$$\left. \times \left[\sum_{\{\sigma_R\} \in P(O\{\alpha_R\} \cup O\{\beta_R\})} \sum_{i=r_L+1}^{r} \sum_{r_L + s_L < \sigma_J < \beta_{Ri}} \left(s_{\beta_{Ri}, J_R} + s_{\beta_{R_i}(\widehat{P}_{\mathcal{I}})} \right) A_{n - r_L - s_L - 2} A(\widehat{P}_{\mathcal{I}}, \{\sigma_R\}, \widehat{n}) \right] \right\}, \tag{3.9}$$

where the kinematic factors with $J = 1$ are shifted, i.e., $s_{\beta_i \widehat{1}}$. In the equation above, we have used $\{\alpha_L\}$, $\{\beta_L\}$ to denote the subsets of α, β at the left hand side and $\{\alpha_R\}$, $\{\beta_R\}$ the subsets of α, β at the right hand side respectively. r_L, s_L, r_R and s_R are the numbers of elements in each subset $\{\beta_L\}$, $\{\alpha_L\}$, $\{\beta_R\}$ and $\{\alpha_R\}$ respectively. $\widehat{P}_{\mathcal{I}}$ is the sum of the momenta at the left hand side of a given splitting $\mathcal{I}$. With the general BCJ relation for sub-amplitudes, each term at the last equation is zero, thus we have shown the contribution of part $\mathbb{A}$ is zero. Combining results from part $\mathbb{A}$ and part $\mathbb{B}$ we proved general BCJ relation(2.6).

4 The proof of the explicit minimal-basis expansion of gauge field tree amplitudes

Although the general BCJ relation (2.6) provides a set of constraint equations, which reduces the number of minimal basis from $(n-2)!$ to $(n-3)!$, the explicit expression of other amplitudes by the minimal basis is not manifest. A manifest expression is conjectured in [2]. In this section, we will prove the conjecture explicitly.

The conjectured minimal-basis expansion can be written as[2]

$$A_n(1, \beta_1, ..., \beta_r, 2, \alpha_1, ..., \alpha_{n-r-3}, n) = \sum_{\{\xi\} \in P(\{\beta\} \cup O\{\alpha\})} A_n(1, 2, \{\xi\}, n) \prod_{k=1}^{r} \frac{\mathcal{F}^{\{\beta\}, \{\alpha\}}(2, \{\xi\}, n|k)}{s_{1, \beta_1, ..., \beta_k}}, \quad (4.1)$$

where $P(\{\beta\} \cup O\{\alpha\})$ corresponds to all permutations of $\{\beta\} \cup \{\alpha\}$ that maintain the relative order of the set $\{\alpha\}$. The function $\mathcal{F}^{\{\beta\}, \{\alpha\}}(2, \{\xi\}, n|k)$ is defined as

$$\mathcal{F}^{\{\beta\}, \{\alpha\}}(2, \{\xi\}, n|k) = \left\{ \begin{array}{ll} \sum_{\xi_J > \xi_{\beta_k}} \mathcal{G}(\beta_k, J) & \text{if } \xi_{\beta_{k-1}} < \xi_{\beta_k} \\ -\sum_{\xi_J < \xi_{\beta_k}} \mathcal{G}(\beta_k, J) & \text{if } \xi_{\beta_{k-1}} > \xi_{\beta_k} \end{array} \right\}$$
$$+ \left\{ \begin{array}{ll} s_{1, \beta_1, ..., \beta_k} & \text{if } \xi_{\beta_{k-1}} < \xi_{\beta_k} < \xi_{\beta_{k+1}} \\ -s_{1, \beta_1, ..., \beta_k} & \text{if } \xi_{\beta_{k-1}} > \xi_{\beta_k} > \xi_{\beta k+1} \\ 0 & \text{else} \end{array} \right\}, \quad (4.2)$$

where ξ_J stands for the position of the leg J in the permutation of ξ and we should include $\xi_0 \equiv \alpha_0 \equiv 2$. We define $\xi_{\beta_0} \equiv \infty$ and $\xi_{\beta_{r+1}} \equiv 0$. The function $\mathcal{G}$ is defined by

$$\mathcal{G}(\beta_k, \beta_j) = \left\{ \begin{array}{ll} s_{\beta_k \beta_j} & \text{if } k < j \\ 0 & \text{else} \end{array} \right\}, \quad (4.3)$$

$$\mathcal{G}(\beta_k, \alpha_j) = s_{\beta_k \alpha_j}, \quad (4.4)$$

where $\alpha_0 = 2$, $\alpha_{n-r-2} = n$, $\xi_2 = \xi_{\alpha_0} \equiv 0$, $\xi_n = \xi_{\alpha_{n-r-2}} \equiv n - 2$.

In this formula, the amplitude with $\beta_1, \beta_2, ..., \beta_r$ between 1 and 2 are expressed by the basis amplitudes with no β between 1 and 2. Thus the formula (4.1) expresses any amplitude by $(n - 3)!$ independent amplitudes with fixed positions of $1, 2, n$ explicitly. We will discuss and prove the minimal-basis expansion (4.1) in the following subsections.

4.1 The properties of the function $\mathcal{F}$

Before we prove the minimal-basis expression (4.1), let us have a look at some useful properties of the function $\mathcal{F}$. It is worth to have some remarks from the definition (4.2). First let us notice that the function has two groups of parameters. The first group of parameters is the up-index $\{\beta\}, \{\alpha\}$, which provides the ordering information of amplitude for which we need to expand into basis. The second group of parameters is $(2, \{\xi\}, n)$, which fixes the particular amplitude of the basis, and the number k which tells us that it is the k-th kinematic factor coming from the k-th element β_k of up-index. The second point we need to notice is that for the k-th kinematic factor, the key information we need is the permutation ξ and the relative ordering between β_k and $\beta_j, j \geq k - 1$. If these information is same, we may get same k-th kinematic factor.

Having observed above points, let us consider the minimal-basis expansion of following two amplitudes. The first one is $A_n(1, \beta_1, ..., \beta_r, 2, \alpha_1, ..., \alpha_{n-r-3}, n)$ given by (4.1). The second one is following expansion

$$A_n(1, \beta_1, ..., \beta_p, 2, \gamma_1, ..., \gamma_{n-p-3}, n)$$
$$= \sum_{\xi \in P(\{\beta_1,...,\beta_p\} \cup O\{\gamma_1,...,\gamma_{n-p-3}\})} A_n(1, 2, \{\xi\}, n) \prod_{k=1}^{p} \frac{\mathcal{F}^{\{\beta_1,...,\beta_p\},\{\gamma_1,...,\gamma_{n-p-3}\}}(2, \{\xi\}, n|k)}{s_{1,\beta_1,...,\beta_p}}, \qquad (4.5)$$

where $p \leq r$ and $\{\gamma_1, ..., \gamma_{n-p-3}\} \in P(O\{\alpha\} \cup O\{\beta_{p+1}, ..., \beta_r\})$, which give the relation between these two would-be-expanded amplitudes.

For $p < r$, from the definition of $\mathcal{F}$ (4.2) we can see

$$F^{\{\beta_1,...,\beta_p\},\{\gamma_1,...,\gamma_{n-p-3}\}}(2, \{\xi\}, n|k) = \mathcal{F}^{\{\beta_1,...,\beta_r\},\{\alpha_1,...,\alpha_{n-r-3}\}}(2, \{\xi\}, n|k), \quad (1 \leq k < p), \qquad (4.6)$$

which is obvious from the definition of (4.2). The boundary case $k = p$ is more complicated and is given by

$$\mathcal{F}^{\{\beta_1,...,\beta_p\},\{\gamma\}}(2, \{\xi\}, n|p) = \begin{cases} \mathcal{F}^{\{\beta_1,...,\beta_r\},\{\alpha_1,...,\alpha_{n-r-3}\}}(2, \{\xi\}, n|p) \\ \mathcal{F}^{\{\beta_1,...,\beta_r\},\{\alpha_1,...,\alpha_{n-r-3}\}}(2, \{\xi\}, n|p) - s_{1\beta_1,...,\beta_p}, \ \xi_{p-1} < \xi_p < \xi_{p+1} \\ \mathcal{F}^{\{\beta_1,...,\beta_r\},\{\alpha_1,...,\alpha_{n-r-3}\}}(2, \{\xi\}, n|p) - s_{1\beta_1,...,\beta_p}, \ \xi_{p-1} > \xi_p, \xi_{p+1} > \xi_p \end{cases} \qquad (4.7)$$

according to different relative orderings among $\xi_{p-1}, \xi_p, \xi_{p+1}$.

Now we consider a given permutation ξ in which the relative order of beta is $\beta_l, \beta_{l+1}, ..., \beta_r$ for a given $1 \leq l < r$, i.e, we have $\xi_{\beta_l} < \xi_{\beta_{l+1}} < ... < \xi_{\beta_r}$. Furthermore we assume $\xi_{\beta_{l-1}} > \xi_{\beta_l}$. With these orderings, following equations can be seen from the definition of $\mathcal{F}$ given by (4.2):

$$\mathcal{F}^{\{\beta_1,...,\beta_r\},\{\alpha\}}(2, \{\xi\}, n|k) = \begin{cases} \sum_{\xi_J > \xi_{\beta_k}} \mathcal{G}(\beta_k, J) + s_{1,\beta_1,...,\beta_k} = - \sum_{\xi_J < \xi_{\beta_k}} \mathcal{G}(\beta_k, J) + s_{1,\beta_1,...,\beta_{k-1}} \ (l < k < r) \\ \sum_{\xi_J > \xi_{\beta_k}} \mathcal{G}(\beta_k, J) = - \sum_{\xi_J < \xi_{\beta_k}} \mathcal{G}(\beta_k, J) - s_{\beta_r 1} - \sum_{j=1}^{r-1} s_{\beta_r \beta_j} \qquad (k = r) \\ - \sum_{\xi_J < \xi_{\beta_k}} \mathcal{G}(\beta_k, J) \qquad (k = l) \end{cases} \qquad (4.8)$$

where we have used the momentum conservation for the first and second lines and the rewriting $s_{1,\beta_1,...,\beta_k} = s_{1,\beta_1,...,\beta_{k-1}} + s_{1\beta_k} + \sum_{i=1}^{k-1} s_{\beta_i \beta_k}$.

The last case we want to discuss is the case with $\xi_{r-1} > \xi_r$. Then by the definition (4.2) we obtain

$$\mathcal{F}^{\{\beta_1,...,\beta_r\},\{\alpha\}}(2, \{\xi\}, n|k = r) = - \sum_{\xi_J < \xi_{\beta_r}} \mathcal{G}(\beta_r, J) - s_{1,\beta_1,...,\beta_r}. \qquad (4.9)$$

Properties (4.6),(4.7),(4.8) and (4.9) are all we need for our proof. It can be briefly seen as following. With the general BCJ relation(2.6), we can express any amplitude with $\beta_1, ..., \beta_r$ between 1 and 2 by amplitudes with less βs between 1 and 2. Then we use minimal-basis expansion to express these amplitudes with less βs between 1 and 2 by those with all βs between 2 and n. For an given amplitude belongs to

the minimal basis, there are several contributions to the coefficient, thus we need to use properties of $\mathcal{F}$ to combine them together. By this way we can prove the minimal basis expansion of amplitude with r βs between 1 and 2 by induction. In the next subsection, we will use some examples to demonstrate this pattern.

4.2 Examples

In this subsection, we will give some examples to see the explicit minimal basis expansion of amplitudes. The first and the simplest example is the minimal-basis expansion of amplitudes with only one element in $\{\beta\}$. The second one is the case with two elements in $\{\beta\}$, the third one is the case with three elements in $\{\beta\}$. Through these examples, the idea of general proof will be more clear.

4.2.1 Only one element in $\{\beta\}$

If there is only one element in $\{\beta\}$, with the general formula of BCJ relation (2.6) we can express the amplitude with β_1 between 1 and 2 by those with β_1 between 2 and n immediately as[4]

$$
A_n(1, \beta_1, 2, \alpha_1, ..., \alpha_{n-4}, n)
$$

$$
= - \sum_{\{\xi\} \in P(O\{\beta_1\} \cup O\{\alpha\})} \frac{\left(s_{1\beta_1} + \sum_{\xi_J < \xi_{\beta_1}} s_{\beta_1 J} \right)}{s_{1\beta_1}} A(1, 2, \{\xi\}, n)
$$

$$
= \sum_{\{\xi\} \in P(\{\beta_1\} \cup O\{\alpha\})} \frac{\mathcal{F}^{\{\beta_1\},\{\alpha\}}(2, \{\xi\}, n|1)}{s_{1\beta_1}} A_n(1, 2, \{\xi\}, n), \tag{4.10}
$$

where we have used the definition (4.2) with the relative ordering $\xi_{\beta_0} = \infty > \xi_{\beta_1} > \xi_{\beta_{r+1}} = 0$. Since there is only one β, the ordered set of permutations $P(O\{\beta_1\} \cup O\{\alpha\})$ becomes the partially ordered set of permutation $P(\{\beta_1\} \cup O\{\alpha\})$ that maintain the order of the $\{\alpha\}$ elements. It is the same result with the minimal-basis expression of amplitudes with only one element in $\{\beta\}$[2]. This example is nothing but the fundamental BCJ relation.

4.2.2 Two elements in $\{\beta\}$

The amplitude with two elements in $\{\beta\}$ is the first nontrivial example. If there are two elements in the set $\{\beta\}$, from the general formula of BCJ relation (2.6), we can see that

$$
A_n(1, \beta_1, \beta_2, 2, \alpha_1, ..., \alpha_{n-5}, n)
$$

$$
= - \sum_{\{\widetilde{\xi}\} \in P(O\{\alpha\} \cup O\{\beta_2\})} \frac{\left(s_{1\beta_1} + s_{1\beta_2} + s_{\beta_2\beta_1} + \sum_{\widetilde{\xi}_J < \widetilde{\xi}_{\beta_2}} s_{\beta_2 J} \right)}{s_{1\beta_1\beta_2}} A_n(1, \beta_1, 2, \{\widetilde{\xi}\}, n)
$$

---

[4]Please remember the convention that $\xi_0 = 2 = \alpha_0$ should be included in the sum over ξ. This convention will be used for all calculations later.

$$- \sum_{\{\xi\} \in P(O\{\alpha\} \cup O\{\beta_1, \beta_2\})} \frac{\left(s_{1\beta_1} + s_{1\beta_2} + \sum_{i=1}^{2} \sum_{\xi_J < \xi_{\beta_i}} s_{\beta_i J} \right)}{s_{1\beta_1\beta_2}} A(1, 2, \{\xi\}, n)$$

$$= - \sum_{\{\widetilde{\xi}\} \in P(O\{\alpha\} \cup O\{\beta_2\})} \frac{\left(s_{1,\beta_1,\beta_2} + \sum_{\widetilde{\xi}_J < \widetilde{\xi}_{\beta_2}} \mathcal{G}(\beta_2, J) \right)}{s_{1\beta_1\beta_2}} \sum_{\{\xi\} \in P(\{\beta_1\} \cup O\{\widetilde{\xi}\})} \frac{\mathcal{F}^{\{\beta_1\}, \{\widetilde{\xi}\}}(2, \{\xi\}, n|1)}{s_{1\beta_1}} A_n(1, 2, \{\xi\}, n)$$

$$- \sum_{\{\xi\} \in P(O\{\alpha\} \cup O\{\beta_1, \beta_2\})} \frac{\left(s_{1,\beta_1,\beta_2} + \sum_{i=1}^{2} \sum_{\xi_J < \xi_{\beta_i}} \mathcal{G}(\beta_i, J) \right)}{s_{1\beta_1\beta_2}} A(1, 2, \{\xi\}, n), \tag{4.11}$$

where we have used the minimal-basis expansion for $A_n(1, \{\beta_1\}, 2, \{\widetilde{\xi}\}, n)$. For the double sum in the first line, it is easy to see that double sum $\sum_{\{\widetilde{\xi}\} \in P(O\{\alpha\} \cup O\{\beta_2\})} \sum_{\{\xi\} \in P(\{\beta_1\} \cup O\{\widetilde{\xi}\})}$ can be written as a single sum $\sum_{\{\xi\} \in P(O\{\alpha\} \cup \{\beta_1, \beta_2\})}$. This single sum can be written into following two sums: (1) case $\mathbb{A} \equiv \sum_{\{\xi\} \in P(O\{\alpha\} \cup O\{\beta_1, \beta_2\})}$; (2) case $\mathbb{B} \equiv \sum_{\{\xi\} \in P(O\{\alpha\} \cup O\{\beta_2, \beta_1\})}$. Case $\mathbb{A}$ needs to combine with the second line of Eq. (4.11) while the case $\mathbb{B}$ is independent itself.

Let us start from the case $\mathbb{B}$ with the ordering $\xi_{\beta_1} > \xi_{\beta_2}$. From the definition of $\mathcal{G}$, we know that $\mathcal{G}(\beta_2, J)$ is independent of the position of β_1, thus the factor $- \left(s_{1,\beta_1,\beta_2} + \sum_{\widetilde{\xi}_J < \widetilde{\xi}_{\beta_2}} \mathcal{G}(\beta_2, J) \right)$ is nothing, but $\mathcal{F}^{\{\beta\}, \{\alpha\}}(2, \{\xi\}, n|2)$. Similarly by using the properties (4.7), the second factor $F^{\{\beta_1\}, \{\widetilde{\xi}\}}(2, \{\gamma\}, n|1) = F^{\{\beta_1, \beta_2\}, \{\alpha\}}(2, \{\xi\}, n|1)$. Combining all together we see that the contribution of case $\mathbb{B}$ is

$$\sum_{\{\xi\} \in P(O\{\alpha\} \cup O\{\beta_2, \beta_1\})} \frac{\mathcal{F}^{\{\beta\}, \{\alpha\}}(2, \{\xi\}, n|2)}{s_{1\beta_1\beta_2}} \frac{\mathcal{F}^{\{\beta\}, \{\alpha\}}(2, \{\xi\}, n|1)}{s_{1\beta_1}} A_n(1, 2, \{\xi\}, n). \tag{4.12}$$

For the ordering $\xi_{\beta_1} < \xi_{\beta_2}$, the contribution is given by

$$- \sum_{\{\xi\} \in P(O\{\alpha\} \cup O\{\beta_1, \beta_2\})} \left[\frac{\left(-\mathcal{F}^{\{\beta\}, \{\alpha\}}(2, \{\xi\}, n|2) + s_{1\beta_1} \right)}{s_{1\beta_1\beta_2}} \frac{\left(\mathcal{F}^{\{\beta\}, \{\alpha\}}(2, \{\xi\}, n|1) - s_{1\beta_1} \right)}{s_{1\beta_1}} \right.$$

$$\left. + \frac{\left(-\mathcal{F}^{\{\beta\}, \{\alpha\}}(2, \{\xi\}, n|2) - \mathcal{F}^{\{\beta\}, \{\alpha\}}(2, \{\xi\}, n|1) + s_{1\beta_1} \right)}{s_{1\beta_1\beta_2}} \right] A_n(1, 2, \{\xi\}, n), \tag{4.13}$$

where we have used the properties (4.7), (4.8) and the definition of $\mathcal{G}$ again. The coefficients of amplitudes with less than two $\mathcal{F}$s cancel out. Only the term with two $\mathcal{F}$s is left

$$\sum_{\{\xi\} \in P(O\{\alpha\} \cup O\{\beta_1, \beta_2\})} \frac{\mathcal{F}^{\{\beta\}, \{\alpha\}}(2, \{\xi\}, n|2)}{s_{1\beta_1\beta_2}} \frac{\mathcal{F}^{\{\beta\}, \{\alpha\}}(2, \{\xi\}, n|1)}{s_{1\beta_1}} A_n(1, 2, \{\xi\}, n). \tag{4.14}$$

Combining results given in (4.12) and (4.14), we see that for both ordering of β_1, β_2, coefficients are just those given in the minimal-basis expansion (4.1), thus we have given the proof of minimal-basis expansion with two βs.

4.2.3 Three elements in $\{\beta\}$

The next nontrivial example is the minimal-basis expansion with three elements in $\{\beta\}$. From the general formula of BCJ relation (2.6), we can express the amplitudes with three βs by amplitudes with βs less than three as

$$
A_n(1, \beta_1, \beta_2, \beta_3, 2, \alpha_1, ..., \alpha_{n-6}, n)
$$

$$
= - \sum_{\{\xi\} \in P(O\{\alpha\} \cup O\{\beta_3\})} \frac{s_{1,\beta_1,\beta_2,\beta_3} + \sum\limits_{\xi_J < \xi_{\beta_3}} s_{\beta_3 J}}{s_{1,\beta_1,\beta_2,\beta_3}} A_n(1, \beta_1, \beta_2, 2, \{\xi\}, n)
$$

$$
- \sum_{\{\xi\} \in P(O\{\alpha\} \cup O\{\beta_2, \beta_3\})} \frac{s_{1\beta_1} + s_{1\beta_2} + s_{\beta_2 \beta_1} + s_{1\beta_3} + s_{\beta_3 \beta_1} + \sum\limits_{i=2}^{3} \sum\limits_{\xi_J < \xi_{\beta_i}} s_{\beta_i J}}{s_{1,\beta_1,\beta_2,\beta_3}} A_n(1, \beta_1, 2, \{\xi\}, n)
$$

$$
- \sum_{\{\xi\} \in P(O\{\alpha\} \cup O\{\beta_1, \beta_2, \beta_3\})} \frac{s_{1\beta_1} + s_{1\beta_2} + s_{1\beta_3} + \sum\limits_{i=1}^{3} \sum\limits_{\xi_J < \xi_{\beta_i}} s_{\beta_i J}}{s_{1,\beta_1,\beta_2,\beta_3}} A_n(1, 2, \{\xi\}, n). \tag{4.15}
$$

With the definition of $\mathcal{G}$ and the minimal-basis expansion with βs less than three, the amplitude becomes

$$
A_n(1, \beta_1, \beta_2, \beta_3, 2, \alpha_1, ..., \alpha_{n-6}, n)
$$

$$
= - \sum_{\{\xi\} \in P(O\{\alpha\} \cup O\{\beta_3\})} \frac{s_{1,\beta_1,\beta_2,\beta_3} + \sum\limits_{\xi_J < \xi_{\beta_3}} \mathcal{G}(\beta_3, J)}{s_{1,\beta_1,\beta_2,\beta_3}}
$$

$$
\times \sum_{\{\gamma\} \in P(\{\beta_1, \beta_2\} \cup O\{\xi\})} \frac{\mathcal{F}^{\{\beta_1, \beta_2\}, \{\xi\}}(2, \{\gamma\}, n|2)}{s_{1,\beta_1,\beta_2}} \frac{\mathcal{F}^{\{\beta_1, \beta_2\}, \{\xi\}}(2, \{\gamma\}, n|1)}{s_{1\beta_1}} A_n(1, 2, \{\gamma\}, n)
$$

$$
- \sum_{\{\xi\} \in OP(O\{\alpha\} \cup O\{\beta_2, \beta_3\})} \frac{s_{1,\beta_1,\beta_2,\beta_3} + \sum\limits_{i=2}^{3} \sum\limits_{\xi_J < \xi_{\beta_i}} \mathcal{G}(\beta_i, J)}{s_{1,\beta_1,\beta_2,\beta_3}} \sum_{\{\gamma\} \in P(\{\beta_1\} \cup O\{\xi\})} \frac{\mathcal{F}^{\{\beta_1\}, \{\xi\}}(2, \{\gamma\}, n|1)}{s_{1,\beta_1}} A_n(1, 2, \{\gamma\}, n)
$$

$$
- \sum_{\{\xi\} \in P(O\{\alpha\} \cup O\{\beta_1, \beta_2, \beta_3\})} \frac{s_{1,\beta_1,\beta_2,\beta_3} + \sum\limits_{i=1}^{3} \sum\limits_{\xi_J < \xi_{\beta_i}} \mathcal{G}(\beta_i, J)}{s_{1,\beta_1,\beta_2,\beta_3}} A_n(1, 2, \{\xi\}, n). \tag{4.16}
$$

It can be seen that the first term gives $3! = 6$ orderings among β_i's, while the second term gives only 3 orderings and the third term, only one.

To consider all these orderings, we classify them into following three categories. The first one is $\xi_{\beta_2} > \xi_{\beta_3}$. The second case is $\xi_{\beta_2} < \xi_{\beta_3}$ but $\xi_{\beta_1} > \xi_{\beta_2}$. The third one is $\xi_{\beta_1} < \xi_{\beta_2} < \xi_{\beta_3}$. We will discuss these three categories one by one.

Case $\xi_{\beta_2} > \xi_{\beta_3}$: For this case, the second and the third terms in (4.16) give no contributions. With the properties of $\mathcal{F}$ and the definition of $\mathcal{G}$, the first term of (4.16) gives

$$\sum_{\{\xi\}\in P(\{\beta_1,\beta_2,\beta_3\}\cup O\{\alpha\}|\xi_{\beta_2}>\xi_{\beta_3})} \frac{\mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|3)}{s_{1,\beta_1,\beta_2,\beta_3}} \frac{\mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|2)}{s_{1,\beta_1,\beta_2}} \frac{\mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|1)}{s_{1\beta_1}}$$
$$\times A_n(1,2,\{\xi\},n). \tag{4.17}$$

This gives the part with $\xi_{\beta_2} > \xi_{\beta_3}$ in the minimal-basis expansion.

Case $\xi_{\beta_2} < \xi_{\beta_3}$ and $\xi_{\beta_1} > \xi_{\beta_2}$: For this case, the third term of (4.16) has no contribution. The combination of first and the second terms of (4.16) gives

$$-\sum_{\xi\in P(\{\beta_1\}\cup O\{\alpha\}\cup O\{\beta_2,\beta_3\}|\xi_{\beta_1}>\xi_{\beta_2})}\left[\frac{\left(-\mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|3)+s_{1,\beta_1,\beta_2}\right)}{s_{1,\beta_1,\beta_2,\beta_3}}\frac{\left(\mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|2)-s_{1,\beta_1,\beta_2}\right)}{s_{1,\beta_1,\beta_2}}\right.$$
$$\left.+\frac{\left(-\mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|3)-\mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|2)+s_{1,\beta_1,\beta_2}\right)}{s_{1,\beta_1,\beta_2,\beta_3}}\right]\frac{\mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|1)}{s_{1\beta_1}}A_n(1,2,\{\xi\},n),$$
$$\tag{4.18}$$

where we have used several properties of $\mathcal{F}$ presented in section 4.1 and the definition of $\mathcal{G}$. In the above expression, all the terms with $\mathcal{F}$s less than three cancel out and only term with three $\mathcal{F}$s is left

$$\sum_{\xi\in P(\{\beta_1\}\cup O\{\alpha\}\cup O\{\beta_2,\beta_3\}|\xi_{\beta_1}>\xi_{\beta_2})} \frac{\mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|3)}{s_{1,\beta_1,\beta_2,\beta_3}}\frac{\mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|2)}{s_{1,\beta_1,\beta_2}}$$
$$\times\frac{\mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|1)}{s_{1,\beta_1}}\times A_n(1,2,\{\xi\},n). \tag{4.19}$$

Thus it gives the contribution of the permutations with $\xi_{\beta_2} < \xi_{\beta_3}$ but $\xi_{\beta_1} > \xi_{\beta_2}$ in the minimal-basis expansion with three βs.

Case $\xi_{\beta_1} < \xi_{\beta_2} < \xi_{\beta_3}$: For this one, all three terms contribute and we need to sum up. Using the properties of $\mathcal{F}$ and the definition of $\mathcal{G}$ again, we have

$$-\sum_{\{\xi\}\in P(O\{\alpha\}\cup O\{\beta_1,\beta_2,\beta_3\})}\left[\frac{\left(-\mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|3)+s_{1,\beta_1,\beta_2}\right)}{s_{1,\beta_1,\beta_2,\beta_3}}\frac{\left(\mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|2)-s_{1,\beta_1,\beta_2}\right)}{s_{1,\beta_1,\beta_2}}\frac{\mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|1)}{s_{1,\beta_1}}\right.$$
$$+\frac{\left(-\mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|3)-\mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|2)+s_{1,\beta_1,\beta_2}+s_{1,\beta_1}\right)}{s_{1,\beta_1,\beta_2,\beta_3}}\frac{\left(\mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|1)-s_{1\beta_1}\right)}{s_{1\beta_1}}$$
$$\left.+\frac{\left(-\mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|3)-\mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|2)-\mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|1)+s_{1,\beta_1,\beta_2}+s_{1,\beta_1}\right)}{s_{1,\beta_1,\beta_2,\beta_3}}\right]A_n(1,2,\{\xi\},n).$$

$$(4.20)$$

All the terms with $\mathcal{F}$s less than three cancel out and we are left with only term having three $\mathcal{F}$s

$$\sum_{\{\xi\}\in P(O\{\alpha\}\cup O\{\beta_1,\beta_2,\beta_3\})}\frac{\mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|3)}{s_{1,\beta_1,\beta_2,\beta_3}}\frac{\mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|2)}{s_{1,\beta_1,\beta_2}}\frac{\mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|1)}{s_{1,\beta_1}}A_n(1,2,\{\xi\},n)\quad(4.21)$$

Summing over all three cases, we get the minimal-basis expansion with three βs.

4.3 General proof

Having these examples above, we have seen the procedure of our general proof. First we use the general formula of BCJ relation(2.6) to express the amplitude with r β's by other amplitudes with β's less than r. Then using the induction, we can substitute these amplitudes with β's less than r by their explicit minimal-basis expansion. Next we divide orderings of r β's into several cases and using the properties of $\mathcal{F}$ and the definition of $\mathcal{G}$ to combine contributions from various terms. The key of the proof is (1) to show the cancelation of these terms with $\mathcal{F}$'s less than r; (2) the remaining term with r $\mathcal{F}$'s are exact the one given by the conjecture. After summing over all permutations we get the minimal-basis expansion(4.1).

Having above discussions, let us start with following expression obtained by using the general formula of BCJ relation (2.6)

$$A_n(1,\beta_1,...,\beta_r,2,\alpha_1,...,\alpha_{n-r-3},n)$$

$$=-\frac{1}{s_{1,\beta_1,...,\beta_r}}\left[\sum_{\{\xi\}\in P(O\{\alpha\}\cup O\{\beta_r\})}\left(s_{1,\beta_1,...,\beta_r}+\sum_{\xi_J<\xi_{\beta_r}}s_{\beta_r J}\right)A_n(1,\beta_1,...,\beta_{r-1},2,\{\xi\},n)\right.$$

$$+\sum_{\{\xi\}\in P(O\{\alpha\}\cup O\{\beta_{r-1},\beta_r\})}\left(s_{1,\beta_1,...,\beta_{r-1}}+s_{\beta_r\beta_{r-2}}+...+s_{\beta_r\beta_1}+s_{\beta_r 1}+\sum_{i=r-1}^{r}\sum_{\xi_J<\xi_{\beta_i}}s_{\beta_i J}\right)$$

$$\times A_n(1,\beta_1,...,\beta_{r-2},2,\{\xi\},n)$$

$$+...$$

$$\left.+\sum_{\{\xi\}\in P(O\{\alpha\}\cup O\{\beta\})}\left(s_{1\beta_r}+...+s_{1\beta_1}+\sum_{i=1}^{r}\sum_{\xi_J<\xi_{\beta_i}}s_{\beta_i J}\right)A_n(1,2,\{\xi\},n)\right].\quad(4.22)$$

Since in the right hand of above equation, all amplitudes have the set β with number less than r, we can substitute the minimal-basis expansions of these amplitudes into (4.22) by induction. Thus we have

$$A_n(1,\beta_1,...,\beta_r,2,\alpha_1,...,\alpha_{n-r-3},n)$$

$$=-\frac{1}{s_{1,\beta_1,...,\beta_r}}\left[\sum_{\{\xi\}\in P(O\{\alpha\}\cup O\{\beta_r\})}\left(s_{1,\beta_1,...,\beta_r}+\sum_{\xi_J<\sigma_{\beta_r}}\mathcal{G}(\beta_r,J)\right)\right.$$

$$\times \sum_{\{\gamma\}\in P(\{\beta_1,...,\beta_{r-1}\}\cup O\{\xi\})} \prod_{k=1}^{r-1} \frac{\mathcal{F}^{\{\beta_1,...,\beta_{r-1}\},\{\xi\}}(2,\{\gamma\},n|k)}{s_{1,\beta_1,...,\beta_k}} A_n(1,2,\{\gamma\},n)$$

$$+ \sum_{\{\xi\}\in P(O\{\alpha\}\cup O\{\beta_{r-1},\beta_r\})} \left(s_{1,\beta_1,...,\beta_r} + \sum_{i=r-1}^{r} \sum_{\xi_J<\xi_{\beta_i}} \mathcal{G}(\beta_i,J) \right)$$

$$\times \sum_{\{\gamma\}\in P(\{\beta_1,...,\beta_{r-2}\}\cup O\{\xi\})} \prod_{k=1}^{r-2} \frac{\mathcal{F}^{\{\beta_1,...,\beta_{r-2}\},\{\xi\}}(2,\{\gamma\},n|k)}{s_{1,\beta_1,...,\beta_k}} A_n(1,2,\{\gamma\},n)$$

$$+ ...$$

$$+ \sum_{\{\xi\}\in P(O\{\alpha\}\cup O\{\beta_1,...,\beta_r\})} \left(s_{1,\beta_1,...,\beta_r} + \sum_{i=1}^{r} \sum_{\xi_J<\xi_{\beta_i}} \mathcal{G}(\beta_i,J) \right) A_n(1,2,\{\xi\},n) \Bigg]. \tag{4.23}$$

As we have seen in previous examples, the first term has $r!$ ordering and the second one, $(r-1)!$ and so on, until the last one, only one ordering among r β's. We divide all orderings to r cases. The first case is the ordering with $\xi_{r-1} > \xi_r$, which has contribution from first term only and is given by

$$\sum_{\xi\in P(\{\beta_1,...,\beta_{r-1},\beta_r\}\cup O\{\alpha\}|\xi_{r-1}>\xi_r)} \prod_{k=1}^{r} \frac{\mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|k)}{s_{1,\beta_1,...,\beta_k}} A_n(1,2,\{\xi\},n), \tag{4.24}$$

where we have already used (4.9).

Other orderings can be characterized with $\xi_{\beta_l} < \xi_{\beta_{l+1}} < ... < \xi_{\beta_r}$ but $\xi_{\beta_{l-1}} > \xi_{\beta_l}$, where the value of l can be any integer between 1 and r. For given l, only first $r-l+1$ terms in (4.23) give nonzero contributions. With the properties of $\mathcal{F}$ (especially (4.8)) and the definition of $\mathcal{G}$, the terms with this ordering can be expressed by $\mathcal{F}^{\{\beta\},\{\alpha\}}$ as following

$$- \sum_{\{\xi\}\in P(\{\beta_1,...,\beta_{l-1}\}\cup O\{\alpha\}\cup O\{\beta_l,...,\beta_r\}|\xi_{\beta_{l-1}}>\xi_{\beta_l})} \frac{1}{s_{1,\beta_1,...,\beta_r} s_{1,\beta_1,...,\beta_{r-1}} \cdots s_{1,\beta_1}}$$

$$\times \Bigg\{ \left(-\mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|r) + s_{1,\beta_1,...,\beta_{r-1}} \right) \left(\mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|r-1) - s_{1,\beta_1,...,\beta_{r-1}} \right)$$

$$\times \mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|r-2) \times ... \times \mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|1)$$

$$+ \sum_{k=l+1}^{r-1} \Bigg[\left(-\mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|r) - \mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|r-1) - ... - \mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|k) \right.$$

$$+ s_{1,\beta_1,...,\beta_{r-1}} + s_{1,\beta_1,...,\beta_{r-2}} + ... + s_{1,\beta_1,...,\beta_{k-1}} \Bigg) \left(\mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|k-1) - s_{1,\beta_1,...,\beta_{k-1}} \right)$$

$$\times \mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|k-2) \times ... \times \mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|1) s_{1,\beta_1,...,\beta_{r-1}} \cdots s_{1,\beta_1,...,\beta_k} \Bigg]$$

$$+ \left(-\mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|r) - \mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|r-1) - ... - \mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|l) \right)$$

$$+ \, s_{1,\beta_1,\ldots,\beta_{r-1}} + s_{1,\beta_1,\ldots,\beta_{r-2}} + \ldots + s_{1,\beta_1,\ldots,\beta_l} \Big)$$

$$\times \mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|l-1) \times \ldots \times \mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|1) s_{1,\beta_1,\ldots,\beta_{r-1}}\cdots s_{1,\beta_1,\ldots,\beta_l} \Bigg\}. \tag{4.25}$$

Eq (4.25) is complicated, so we will discuss its structure inside the big curly braked in details:

- (1) First it is easy to see that there are $r - l + 1$ terms and each term is given by the multiplication of various dynamical factor s and $\mathcal{F}$. For the k-th term, there are $r + 1 - k$ factors containing $\mathcal{F}$ and $k - 1$ kinematic factors given by $s_{1\ldots\beta}$.

- (2) The structure of kinematic factors is following. For the first term, it is 1. The second term has factor $s_{1,\beta_1,\ldots,\beta_{r-1}}$ and the third term, $s_{1,\beta_1,\ldots,\beta_{r-1}} s_{1,\beta_1,\ldots,\beta_{r-2}}$. So for k-th term, it is $\prod_{i=1}^{k-1} s_{1,\beta_1,\ldots,\beta_{r-i}}$.

- (3) The structure of $\mathcal{F}$ is more complicated. Let us use $\mathcal{F}_t \equiv F^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|t)$, then the first term is $(-\mathcal{F}_r + s)(\mathcal{F}_{r-1} - s) \prod_{i=1}^{r-2} \mathcal{F}_{r-1-i}$. Especially only first two $\mathcal{F}$ combining with proper factor s. For the second term, we merge first two $\mathcal{F}$ factors, i.e., $(-\mathcal{F}_r + s)(\mathcal{F}_{r-1} - s) \to (-\mathcal{F}_r - \mathcal{F}_{r-1} + s + s)$ while changing next pure $\mathcal{F}_{r-2}$ factor into $(\mathcal{F}_{r-2} - s)$. This procedure is used recursively to the third, fourth terms and so on. The only exception is for the last term, we do the merge only, but not the changing.

  Having above pattern, we can see that if we consider the power of $\mathcal{F}$, the first term contains power $r, r-1, r-2$, while the second term, power $r-1, r-2, r-3$. The observation can be summerized by following $(r - l + 1) \times (r - l + 1)$ matrix:

$$\begin{bmatrix} r & r-1 & r-2 & & & & \\ 0 & r-1 & r-2 & r-3 & & & \\ & & r-2 & r-3 & r-4 & & \\ . & . & . & . & . & . & . \\ & & & & l+1 & l & l-1 \\ & & & & & l & l-1 \end{bmatrix}. \tag{4.26}$$

- (4) From previous observation, we see that for the $(r - k + 1)$-th term, the first two nontrivial factors have following structure. The first factor is

$$\Big(-\mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|r) - \mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|r-1) - \ldots - \mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|k)$$

$$+ s_{1,\beta_1,\ldots,\beta_{r-1}} + s_{1,\beta_1,\ldots,\beta_{r-2}} + \ldots + s_{1,\beta_1,\ldots,\beta_{k-1}}\Big), \tag{4.27}$$

while the second factor is

$$\Big(\mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|k-1) - s_{1,\beta_1,\ldots,\beta_{k-1}}\Big) \tag{4.28}$$

Having above structure (4.26), now we can see how to finish our proof. The whole expression has only one term with n factors $\mathcal{F}$, which is given in the first term. It is nothing, but the result we want to prove. Other contributions with less power of $\mathcal{F}$ will cancel each other.

To see the cancelation of various powers, let us start with the power $(r-1)$, for which we need to sum up contributions from the first and second terms. Recalling the factor $s_{1,\beta_1,\ldots,\beta_{r-1}}$ of second term, the contribution of power $r-1$ from the second term is given by (where the factor $\mathcal{F}_{r-2}\ldots\mathcal{F}_1$ is neglected)

$$(-\mathcal{F}_r - \mathcal{F}_{r-1})s_{1,\beta_1,\ldots,\beta_{r-1}} \tag{4.29}$$

while the contribution from the first term,

$$(-\mathcal{F}_r + s_{1,\beta_1,\ldots,\beta_{r-1}})(\mathcal{F}_{r-1} - s_{1,\beta_1,\ldots,\beta_{r-1}}) \rightarrow (\mathcal{F}_r + \mathcal{F}_{r-1})s_{1,\beta_1,\ldots,\beta_{r-1}} \tag{4.30}$$

Next we move to the case with power $r-2$, where the first, second and third terms contribute(where the factor $\mathcal{F}_{r-3}\ldots\mathcal{F}_1$ is neglected):

$$
\begin{aligned}
First: \qquad & -s_{1,\beta_1,\ldots,\beta_{r-1}}^2 \mathcal{F}_{r-2} \\
Second: \qquad & s_{1,\beta_1,\ldots,\beta_{r-1}} \mathcal{F}_{r-2}(s_{1,\beta_1,\ldots,\beta_{r-1}} + s_{1,\beta_1,\ldots,\beta_{r-2}}) \\
& + s_{1,\beta_1,\ldots,\beta_{r-1}}(\mathcal{F}_r + \mathcal{F}_{r-1})s_{1,\beta_1,\ldots,\beta_{r-2}} \\
Third: \qquad & s_{1,\beta_1,\ldots,\beta_{r-1}}s_{1,\beta_1,\ldots,\beta_{r-2}}(-\mathcal{F}_r - \mathcal{F}_{r-1} - \mathcal{F}_{r-2})
\end{aligned}
\tag{4.31}
$$

From above table, it is easy to see the cancelation.

The cases of power $(r-1)$ and $(r-2)$ give, in fact, the general pattern of cancelations. The general form of power $(h+1)$ of $\mathcal{F}$ can be written as

$$f(g,h)\mathcal{F}_g \prod_{i=1}^{h} \mathcal{F}_i, \tag{4.32}$$

where $g \geq h+1$ and $f(g,h)$ is a function of s_{ij}. As seen from the (4.27), (4.28) and (4.31), we should divide the discussion of g to two cases $g = h+1$ and $g > (h+1)$ and for each case, we should carefully deal with the boundary given by the first and last term of (4.25).

The case $g > h+1$: For this case, there are only two terms in (4.25) giving contributions, i.e., the terms with $k = h+1$ and $k = h+2$. The term with $k = h+1$ gives a factor $-s_{1,\beta_1,\ldots,\beta_{r-1}}\ldots s_{1,\beta_1,\ldots,\beta_{h+1}}$ while the term with $k = h+2$ gives a factor $s_{1,\beta_1,\ldots,\beta_{r-1}}\ldots s_{1,\beta_1,\ldots,\beta_{h+1}}$. Thus the sum of these two give $f(g,h) = 0$ for $g > h+1$.

The boundary cases for $g > h+1$ should also be discussed independently. For the upper boundary is nothing, but the one given in (4.31). For the lower boundary, we should consider the case $h = l-1$, $g > l$. In this case, the last term in (4.25) gives a factor $-s_{1,\beta_1,\ldots,\beta_{r-1}}\ldots s_{1,\beta_1,\ldots,\beta_l}$ while the term with $k = l+1$ gives a factor $s_{1,\beta_1,\ldots,\beta_{r-1}}\ldots s_{1,\beta_1,\ldots,\beta_l}$. These two factors cancel with each other. Thus we get $f(g,l-1) = 0$ for $g > l$.

The case $g = h + 1$: There are three terms in (4.25) give contributions. The term $k = g = h + 1$ gives a factor $-s_{1,\beta_1,...,\beta_{r-1}}...s_{1,\beta_1,...,\beta_{h+1}}$. The them $k = g + 1 = h + 2$ gives a factor $(s_{1,\beta_1,...,\beta_{r-1}} + s_{1,\beta_1,...,\beta_{r-2}} + ... + s_{1,\beta_1,...,\beta_{h+2}} + s_{1,\beta_1,...,\beta_{h+1}})s_{1,\beta_1,...,\beta_{r-1}}...s_{1,\beta_1,...,\beta_{h+2}}$. The term $k = g + 2 = h + 3$ gives a factor $-(s_{1,\beta_1,...,\beta_{r-1}} + s_{1,\beta_1,...,\beta_{r-2}} + ... + s_{1,\beta_1,...,\beta_{h+2}})s_{1,\beta_1,...,\beta_{r-1}}...s_{1,\beta_1,...,\beta_{h+2}}$. Thus all three factors cancel out with each other, we have $f(h + 1, h) = 0$.

Now we move to the boundary case. The upper boundary has been given above. For the lower boundary, two cases $g = h + 1 = l$ and $g = h + 1 = l - 1$ should be considered. For case $g = h + 1 = l$, the term with $k = l + 2$ gives a factor $-(s_{1,\beta_1,...,\beta_{r-1}} + s_{1,\beta_1,...,\beta_{r-2}} + ... + s_{1,\beta_1,...,\beta_{l+2}} + s_{1,\beta_1,...,\beta_{l+1}})s_{1,\beta_1,...,\beta_{r-1}}...s_{1,\beta_1,...,\beta_{l+1}}$, while the term with $k = l + 1$ gives a factor $(s_{1,\beta_1,...,\beta_{r-1}} + s_{1,\beta_1,...,\beta_{r-2}} + ... + s_{1,\beta_1,...,\beta_{l+1}} + s_{1,\beta_1,...,\beta_l})s_{1,\beta_1,...,\beta_{r-1}}...s_{1,\beta_1,...,\beta_{l+1}}$ and the last term in (4.25) gives a factor $-s_{1,\beta_1,...,\beta_{r-1}}...s_{1,\beta_1,...,\beta_{l+1}}s_{1,\beta_1,...,\beta_l}$. Summing three terms up, we have $f(l, l - 1) = 0$. For case $g = h + 1 = l - 1$, the term with $k = l + 1$ gives a factor $-(s_{1,\beta_1,...,\beta_{r-1}} + s_{1,\beta_1,...,\beta_{r-2}} + ... + s_{1,\beta_1,...,\beta_{l+1}} + s_{1,\beta_1,...,\beta_l})s_{1,\beta_1,...,\beta_{r-1}}...s_{1,\beta_1,...,\beta_l}$ while the last term in (4.25) gives a factor $(s_{1,\beta_1,...,\beta_{r-1}} + s_{1,\beta_1,...,\beta_{r-2}} + ... + s_{1,\beta_1,...,\beta_{l+1}} + s_{1,\beta_1,...,\beta_l})\, s_{1,\beta_1,...,\beta_{r-1}}...s_{1,\beta_1,...,\beta_l}$ Summing them up we get $f(l - 1, l - 2) = 0$.

Up to now, it is clear that all terms with $\mathcal{F}$s less than r cancel out. Only the term with r $\mathcal{F}$s is left. Thus we obtain

$$\sum_{\{\xi\}\in P(\{\beta_1,...,\beta_{l-1}\}\cup O\{\alpha\}\cup O\{\beta_l,...,\beta_r\}|\xi_{\beta_{l-1}}>\xi_{\beta_l})} \frac{\mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|r)}{s_{1,\beta_1,...,\beta_r}} \frac{\mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|r-1)}{s_{1,\beta_1,...,\beta_{r-1}}}...$$
$$\times \frac{\mathcal{F}^{\{\beta\},\{\alpha\}}(2,\{\xi\},n|1)}{s_{1\beta_1}} \times A_n(1,2,\{\xi\},n). \tag{4.33}$$

After summing over all permutations, we get the expression of explicit minimal-basis expansion (4.1).

5   Summary

In this paper, we have done following two things. First we gave a field theory proof of the general BCJ relation obtained by taking the field theory limit of imaginary part of monodromy relation in string theory. Using this, we proved the explicit minimal-basis expansion of gauge field tree amplitudes which was conjectured in [2].

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